process occurs at time S_n, where

$$S_n \equiv X_1 + X_2 + \cdots + X_n.$$

The resultant counting process $\{N(t), t \geq 0\}$ will be Poisson with rate λ.

2.3 Conditional Distribution of the Arrival Times

Suppose we are told that exactly one event of a Poisson process has taken place by time t, and we are asked to determine the distribution of the time at which the event occurred. Since a Poisson process possesses stationary and independent increments, it seems reasonable that each interval in $[0, t]$ of equal length should have the same probability of containing the event In other words, the time of the event should be uniformly distributed over $[0, t]$. This is easily checked since, for $s \leq t$,

$$\begin{aligned} P\{X_1 < s | N(t) = 1\} &= \frac{P\{X_1 < s, N(t) = 1\}}{P\{N(t) = 1\}} \\ &= \frac{P\{1 \text{ event in } [0, s), 0 \text{ events in } [s, t)\}}{P\{N(t) = 1\}} \\ &= \frac{P\{1 \text{ event in } [0, s)\} P\{0 \text{ events in } [s, t)\}}{P\{N(t) = 1\}} \\ &= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\ &= \frac{s}{t}. \end{aligned}$$

This result may be generalized, but before doing so we need to introduce the concept of order statistics

Let $Y_1, Y_2, \ldots, Y_n$ be n random variables. We say that $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are the order statistics corresponding to $Y_1, Y_2, \ldots, Y_n$ if $Y_{(k)}$ is the kth smallest value among $Y_1, \ldots, Y_n$, $k = 1, 2, \ldots, n$. If the Y_i's are independent identically distributed continuous random variables with probability density f, then the joint density of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is given by

$$f(y_1, y_2, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i), \qquad y_1 < y_2 < \cdots < y_n.$$

The above follows since (i) $(Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)})$ will equal $(y_1, y_2, \ldots, y_n)$ if $(Y_1, Y_2, \ldots, Y_n)$ is equal to any of the $n!$ permutations of $(y_1, y_2, \ldots, y_n)$, and (ii) the probability density that $(Y_1, Y_2, \ldots, Y_n)$ is equal to $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$ is $f(y_{i_1})f(y_{i_2}) \cdots f(y_{i_n}) = \prod_1^n f(y_i)$ when $(y_{i_1}, y_{i_2}, \ldots, y_{i_n})$ is a permutation of $(y_1, y_2, \ldots, y_n)$.

If the Y_i, $i = 1, \ldots, n$, are uniformly distributed over $(0, t)$, then it follows from the above that the joint density function of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is

$$f(y_1, y_2, \ldots, y_n) = \frac{n!}{t^n}, \qquad 0 < y_1 < y_2 < \cdots < y_n < t.$$

We are now ready for the following useful theorem.

THEOREM 2.3.1

Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.

Proof We shall compute the conditional density function of $S_1, \ldots, S_n$ given that $N(t) = n$ So let $0 < t_1 < t_2 < \cdots < t_{n+1} = t$ and let h_i be small enough so that $t_i + h_i < t_{i+1}$, $i = 1, \ldots, n$ Now,

$$P\{t_i \le S_i \le t_i + h_i, i = 1, 2, \ldots, n \mid N(t) = n\}$$

$$= \frac{P\{\text{exactly 1 event in } [t_i, t_i + h_i], i = 1, \ldots, n, \text{no events elsewhere in } [0, t]\}}{P\{N(t) = n\}}$$

$$= \frac{\lambda h_1 e^{-\lambda h_1} \cdots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - h_2 - \cdots - h_n)}}{e^{-\lambda t}(\lambda t)^n / n!}$$

$$= \frac{n!}{t^n} h_1 \cdot h_2 \cdots h_n$$

Hence,

$$\frac{P\{t_i \le S_i \le t_i + h_i, i = 1, 2, \ldots, n \mid N(t) = n\}}{h_1 \, h_2 \cdots h_n} = \frac{n!}{t^n},$$

and by letting the $h_i \to 0$, we obtain that the conditional density of $S_1, \ldots, S_n$ given that $N(t) = n$ is

$$f(t_1, \ldots, t_n) = \frac{n!}{t^n}, \qquad 0 < t_1 < \cdots < t_n,$$

which completes the proof

Remark Intuitively, we usually say that under the condition that n events have occurred in $(0, t)$, the times $S_1, \ldots, S_n$ at which events occur, considered as unordered random variables, are distributed independently and uniformly in the interval $(0, t)$

Example 2.3(a) Suppose that travelers arrive at a train depot in accordance with a Poisson process with rate λ If the train departs at time t, let us compute the expected sum of the waiting times of travelers arriving in $(0, t)$. That is, we want $E[\sum_{i=1}^{N(t)} (t - S_i)]$, where S_i is the arrival time of the ith traveler Conditioning on $N(t)$ yields

$$E\left[\sum_{i=1}^{N(t)} (t - S_i) | N(t) = n\right] = E\left[\sum_{i=1}^{n} (t - S_i) | N(t) = n\right]$$
$$= nt - E\left[\sum_{i=1}^{n} S_i | N(t) = n\right]$$

Now if we let $U_1, \ldots, U_n$ denote a set of n independent uniform $(0, t)$ random variables, then

$$E\left[\sum_{i=1}^{n} S_i | N(t) = n\right] = E\left[\sum_{i=1}^{n} U_{(i)}\right] \quad \text{(by Theorem 2 3.1)}$$
$$= E\left[\sum_{i=1}^{n} U_i\right] \quad \left(\text{since } \sum_1^n U_{(i)} = \sum_1^n U_i\right)$$
$$= \frac{nt}{2}.$$

Hence,

$$E\left[\sum_{1}^{N(t)} (t - S_i) | N(t) = n\right] = nt - \frac{nt}{2} = \frac{nt}{2}$$

and

$$E\left[\sum_{1}^{N(t)} (t - S_i)\right] = \frac{t}{2} E[N(t)] = \frac{\lambda t^2}{2}.$$

As an important application of Theorem 2 3.1 suppose that each event of a Poisson process with rate λ is classified as being either a type-I or type-II event, and suppose that the probability of an event being classified as type-I depends on the time at which it occurs. Specifically, suppose that if an event occurs at time s, then, independently of all else, it is classified as being a type-I event with probability $P(s)$ and a type-II event with probability $1 - P(s)$. By using Theorem 2.3.1 we can prove the following proposition.

PROPOSITION 2.3.2

If $N_i(t)$ represents the number of type-i events that occur by time t, $i = 1, 2$, then $N_1(t)$ and $N_2(t)$ are independent Poisson random variables having respective means λtp and $\lambda t(1 - p)$, where

$$p = \frac{1}{t}\int_0^t P(s)\,ds$$

Proof We compute the joint distribution of $N_1(t)$ and $N_2(t)$ by conditioning on $N(t)$

$$\begin{aligned} P\{N_1(t) = n, N_2(t) = m\} \\ &= \sum_{k=0}^{\infty} P\{N_1(t) = n, N_2(t) = m \mid N(t) = k\}P\{N(t) = k\} \\ &= P\{N_1(t) = n, N_2(t) = m \mid N(t) = n + m\}P\{N(t) = n + m\} \end{aligned}$$

Now consider an arbitrary event that occurred in the interval $[0, t]$ If it had occurred at time s, then the probability that it would be a type-I event would be $P(s)$ Hence, since by Theorem 2 3 1 this event will have occurred at some time uniformly distributed on $(0, t)$, it follows that the probability that it will be a type-I event is

$$p = \frac{1}{t}\int_0^t P(s)\,ds$$

independently of the other events Hence, $P\{N_1(t) = n, N_2(t) = m \mid N(t) = n + m\}$ will just equal the probability of n successes and m failures in $n + m$ independent trials when p is the probability of success on each trial. That is,

$$P\{N_1(t) = n, N_2(t) = m \mid N(t) = n + m\} = \binom{n+m}{n} p^n (1-p)^m$$

Consequently,

$$\begin{aligned} P\{N_1(t) = n, N_2(t) = m\} &= \frac{(n+m)!}{n!\,m!} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\ &= e^{-\lambda tp} \frac{(\lambda tp)^n}{n!} e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^m}{m!}, \end{aligned}$$

which completes the proof

The importance of the above proposition is illustrated by the following example